Indian Statistical Institute
M.Math II Year
Second Semester Examination, 2004-2005
Stochastic Processes-II

Time: 3 hrs Date:12-05-05 Max. Marks : 100

1. a) Suppose $X = \{X_n, \ -\infty < n < \infty\}$ and $Y = \{Y_n, \ -\infty < n < \infty\}$ are mean zero, stationary L^2-processes with $X_n = \sum\limits_{j=-\infty}^{\infty} a_j Y_{n-j}$, the series converging in L^2. Let μ_X and μ_Y be the spectral measures of X and Y respectively on $[-\pi, \ \pi]$. If $\phi(\lambda) = \sum\limits_{j=-\infty}^{\infty} a_j e^{-i\lambda j}$ $\lambda \in [-\pi, \ \pi]$ converges in $L^2(\mu_Y)$, show that

$$\mu_X(E) = \int_E |\phi(\lambda)|^2 d\mu_Y(\lambda).$$

[10]

b) Conversely, suppose that $X = \{X_n, -\infty < n < \infty\}$ is a mean zero, stationary L^2-process whose spectral measure μ_X is given as

$$\mu_X(E) = \int_E |\phi(\lambda)|^2 d\mu(\lambda)$$

for some finite measure μ on $[-\pi, \pi]$ and $\phi \in L^2(\mu)$.

Suppose further that $\phi(\lambda) = \sum\limits_{j=-\infty}^{\infty} a_j e^{-i\lambda j}$ for some complex numbers a_j and the series converges in $L^2(\mu)$. If $|\phi(\lambda)| > 0$ a.e. μ, show that there exists a stationary, mean zero, L^2-process $\{Y_n, \ -\infty < n < \infty\}$ with spectral measure μ such that $X_n = \sum\limits_{j=-\infty}^{\infty} a_j Y_{n-j}$. [10]

2. For each $n \geq 1$, let $\{W_n(t), \ t \in I\!R\}$ be a sequence of mean zero, stationary L^2-process with spectral density given by $f_n(\lambda) = I_{[-n,n]}(\lambda)$. For $g \in L^2(m), \ m$ being Lebesgue measure, let $W_n(g) = \int\limits_{I\!R} g(t) W_n(t) dm(t)$.

a) For each $g \in L^2(m)$, show that there exists an L^2-random variable $W(g)$ such that $\lim\limits_{n\to\infty} W_n(g) = W(g)$ in L^2. Show that $W(g)$ satisfies

$$E(W(g)\overline{W(h)}) = 2\pi \int_{-\infty}^{\infty} g(t)\overline{h(t)} dm(t).$$

[10]

b) Show that there exists a process with orthogonal increments $\{Z(\lambda), -\infty < \lambda < \infty\}$ such that $\int\limits_{I\!R} g(\lambda) dZ(\lambda) = W(g)$ for all $g \in L^2(m)$.

[10]

3. Let (B_t) be a d-dimensional standard BM . Let P_x be the law of $(B_t + x)_{t \geq 0}$ on $(C[0,\infty), \mathcal{C})$. Let $\mathcal{F}_s = \sigma\{B_t,\ t \leq s\}$. Let $B_{s+.} : \Omega \to C[0,\infty)$ be the map $\omega \to (B_{s+t}(\omega))_{t\geq 0}$. Show that the regular conditional distribution of $B_{s+.}$ given $\mathcal{F}_s$ is given by the map $(\omega, A) \to P_{B_s(\omega)}(A)$.

[10]

4. Let (B_t) be a standard one dimensional BM and $M_t = \sup\limits_{s \leq t} B_s$. Show that the joint density of (B_t, M_t) is given by

$$f(x,z) = \begin{cases} 0 & x > z \quad \text{or } z < 0 \\ \sqrt{\frac{2}{\pi}} \frac{2z-x}{t^{3/2}} e^{-\frac{(2z-x)^2}{2t}} & z \geq 0 \quad \text{and } x \leq z \end{cases}$$

Hint: Compute $P\{B_t < z - y,\ M_t \geq z\}$ for $z \geq 0, y \geq 0$. [10]

5. Let (B_t) be a standard one dimensional BM. Show that for any $b \in I\!R$, the random set $\{t :\ B_t = b\}$ is a closed, unbounded, perfect set of Lebesgue measure zero. [15]

6. Let (B_t) be a standard one dimensional BM and for $b > 0$ let τ_b be the first passage time for b. Let $Y_t = B_{t \wedge \tau_b}$. Show that $(Y_t)_{t\geq 0}$ is a Markov process, starting from zero with the transition function $K_2(t,\ x,\ \cdot)$ defined as follows:

$$K_2(t,\ x,\ \cdot) = \delta_x(\cdot) \quad \text{if } t = 0 \ \text{ or } t > 0 \ \text{ and } x \geq b.$$

If $x < b$ and $A \subseteq (-\infty,\ b)$ and $t > 0$, then

$$K_2(t,\ x,\ A) = \int\limits_A \frac{1}{\sqrt{2\pi t}} \left\{ e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(2b-x-y)^2}{2t}} \right\} dy.$$

and $K_2(t,\ x, \{b\}) = P\{M_t \geq b - x\}$. [15]

7. Let $\{\zeta_j\}_{j=1}^{\infty}$ be a sequence of independent and identically distributed r.v's on $(\Omega, \mathcal{F}, \mathcal{P})$. Let $S_0 = 0,\quad S_k = \sum\limits_{j=1}^{k} \zeta_j,\quad k \geq 1$. Let

$$
\begin{aligned}
Y_t &= S_{[t]} + (t - [t])\zeta_{[t]+1} \quad t \geq 0 \\
X_t^n &= \frac{1}{\sigma\sqrt{n}} Y_{nt} \qquad\qquad\quad t \geq 0
\end{aligned}
$$

a) Show that for every $\epsilon > 0,\ T > 0,$

$$
\lim_{\delta \downarrow 0} \sup_{n \geq 1} P \left\{ \max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |X_s^n - X_t^n| > \epsilon \right\} = 0
$$

[You may state, without proof appropriate results about the S_k's that you need]. [9]

b) Show that for $0 \leq t, < \ldots < t_k$, the random vector $(X_{t_1}^n, \ldots X_{t_k}^n)$ converges in distribution to $(B_{t_1}, \ldots B_{t_k})$, where $(B_t)_{t \geq 0}$ is a standard one dimensional Brownian motion. [9]

c) Let P_n be the law of $(X_t^n)_{t \geq 0}$ on $(C[0, \infty), \mathcal{C})$. Deduce from a) and b) that $\{P_n\}$ converges weakly to the Weiner measure. [7]